

The Finsler-like geometry of the (t, x) -conformal deformation of the jet Berwald-Moór metric

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Abstract

The aim of this paper is to develop on the 1-jet space $J^1(\mathbb{R}, M^n)$ the Finsler-like geometry (in the sense of distinguished (d-) connection, d-torsions, d-curvatures and some gravitational-like and electromagnetic-like geometrical models) attached to the (t, x) -conformal deformation of the Berwald-Moór metric.

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1 Introduction

The geometric-physical Berwald-Moór structure ([5], [12], [11]) was intensively investigated by P.K. Rashevski ([18]) and further fundamented and developed by D.G. Pavlov, G.I. Garas'ko and S.S. Kokarev ([15], [16], [8], [17]). At the same time, the physical studies of Asanov ([1]) or Garas'ko and Pavlov ([8]) emphasize the importance of the Finsler geometry characterized by the total equality in rights of all non-isotropic directions in the theory of space-time structure, gravitation and electromagnetism. For such a reason, one underlines the important role played by the Berwald-Moór metric

$$F : TM \rightarrow \mathbb{R}, \quad F(y) = \sqrt[n]{y^1 y^2 \dots y^n}, \quad n \geq 2,$$

whose tangent Finslerian geometry is studied by geometers as Matsumoto and Shimada ([9]) or Balan ([3]). In such a perspective, according to the recent geometric-physical ideas proposed by Garas'ko in [7] and [6], we consider that a Finsler-like geometric-physical study for the (t, x) -conformal deformations of the jet Berwald-Moór structure is required. Consequently, this paper investigates on the 1-jet space $J^1(\mathbb{R}, M^n)$ the Finsler-like geometry (together with a theoretical-geometric gravitational field-like theory) of the (t, x) -conformal deformation of

the Berwald-Moór metric¹

$${}^*F(t, x, y) = e^{\sigma(x)} \sqrt{h^{11}(t)} \cdot [y_1^1 y_1^2 \dots y_1^n]^{\frac{1}{n}}, \quad (1.1)$$

where $\sigma(x)$ is a smooth non-constant function on M^n , $h^{11}(t)$ is the dual of a Riemannian metric $h_{11}(t)$ on \mathbb{R} , and

$$(t, x, y) = (t, x^1, x^2, \dots, x^n, y_1^1, y_1^2, \dots, y_1^n)$$

are the coordinates of the 1-jet space $J^1(\mathbb{R}, M^n)$, which transform by the rules (the Einstein convention of summation is assumed everywhere):

$$\tilde{t} = \tilde{t}(t), \quad \tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{y}_1^i = \frac{\partial \tilde{x}^i}{\partial x^j} \frac{dt}{d\tilde{t}} \cdot y_1^j, \quad (1.2)$$

where $i, j = \overline{1, n}$, $\text{rank}(\partial \tilde{x}^i / \partial x^j) = n$ and $d\tilde{t}/dt \neq 0$. Note that the particular jet Finsler-like geometries (together with their corresponding jet geometrical gravitational field-like theories) of the (t, x) -conformal deformations of the Berwald-Moór metrics of order three and four are now completely developed in the papers [13] and [14].

Based on the geometrical ideas promoted by Miron and Anastasiei in the classical Lagrangian geometry of tangent bundles ([10]), together with those used by Asanov in the geometry of 1-jet spaces ([2]), the differential geometry (in the sense of d-connections, d-torsions, d-curvatures, gravitational and electromagnetic geometrical theories) produced by an arbitrary jet rheonomic Lagrangian function $L : J^1(\mathbb{R}, M^n) \rightarrow \mathbb{R}$ is now exposed in the monograph [4]. In what follows, we apply the general jet geometrical results from book [4] to the (t, x) -conformal deformed jet Berwald-Moór metric (1.1).

2 The canonical nonlinear connection

Let us rewrite the (t, x) -conformal deformed jet Berwald-Moór metric (1.1) in the form

$${}^*F(t, x, y) = e^{\sigma(x)} \sqrt{h^{11}(t)} \cdot [G_{1[n]}(y)]^{1/n},$$

where

$$G_{1[n]}(y) = y_1^1 y_1^2 \dots y_1^n.$$

Hereinafter, the *fundamental metrical d-tensor* produced by the metric (1.1) is given by the formula² (see [4])

$${}^*g_{ij}(t, x, y) \stackrel{\text{def}}{=} \frac{h_{11}(t)}{2} \frac{\partial^2 {}^*F^2}{\partial y_1^i \partial y_1^j} \Rightarrow$$

¹We assume that we have $y_1^1 y_1^2 \dots y_1^n > 0$. This is a domain of existence where we can y -differentiate the Finsler-like function ${}^*F(t, x, y)$.

²Throughout this paper the Latin letters i, j, k, m, r, \dots take values in the set $\{1, 2, \dots, n\}$.

$${}^*g_{ij}(t, x, y) := {}^*g_{ij}(x, y) = \frac{e^{2\sigma(x)}}{n} \left(\frac{2}{n} - \delta_{ij} \right) \frac{G_{1[n]}^{2/n}}{y_1^i y_1^j}, \quad (2.1)$$

where we have no sum by i or j . Moreover, the matrix ${}^*g = ({}^*g_{ij})$ admits the inverse ${}^*g^{-1} = ({}^*g^{jk})$, whose entries are

$${}^*g^{jk} = e^{-2\sigma(x)} (2 - n\delta^{jk}) G_{1[n]}^{-2/n} y_1^j y_1^k \text{ (no sum by } j \text{ or } k). \quad (2.2)$$

Let us consider that the Christoffel symbol of the Riemannian metric h_{11} on \mathbb{R} is

$$\kappa_{11}^1 = \frac{h^{11}}{2} \frac{dh_{11}}{dt},$$

where $h^{11} = 1/h_{11} > 0$. Then, using a general formula from [4] and taking into account that we have

$$\frac{\partial G_{1[n]}}{\partial y_1^i} = \frac{G_{1[n]}}{y_1^i},$$

we find the following geometrical result:

Proposition 2.1 *For the (t, x) -conformal deformed Berwald-Moór metric (1.1), the energy action functional*

$$\mathbf{E}^*(t, x(t)) = \int_a^b {}^*F^2(t, x, y) \sqrt{h_{11}} dt = \int_a^b e^{2\sigma(x)} [y_1^1 y_1^2 \dots y_1^n]^{2/n} \cdot h^{11} \sqrt{h_{11}} dt,$$

where $y = dx/dt$, produces on the 1-jet space $J^1(\mathbb{R}, M^n)$ the **canonical non-linear connection**

$$\Gamma^* = \left(M_{(1)1}^{(i)} = -\kappa_{11}^1 y_1^i, \quad N_{(1)j}^{(i)} = n\sigma_i y_1^i \delta_j^i \right), \quad (2.3)$$

where

$$\sigma_i = \frac{\partial \sigma}{\partial x^i}.$$

Proof. For the energy action functional \mathbf{E}^* , the associated Euler-Lagrange equations can be written in the equivalent form (see [4])

$$\frac{d^2 x^i}{dt^2} + 2H_{(1)1}^{(i)}(t, x^k, y_1^k) + 2G_{(1)1}^{(i)}(t, x^k, y_1^k) = 0, \quad (2.4)$$

where the local components

$$H_{(1)1}^{(i)} \stackrel{def}{=} -\frac{1}{2} \kappa_{11}^1(t) y_1^i$$

and

$$\begin{aligned} G_{(1)1}^{(i)} &\stackrel{def}{=} \frac{h_{11} {}^*g^{ip}}{4} \left[\frac{\partial^2 {}^*F^2}{\partial x^r \partial y_1^p} y_1^r - \frac{\partial {}^*F^2}{\partial x^p} + \frac{\partial^2 {}^*F^2}{\partial t \partial y_1^p} + \right. \\ &\quad \left. + \frac{\partial {}^*F^2}{\partial y_1^p} \kappa_{11}^1(t) + 2h^{11} \kappa_{11}^1 {}^*g_{pr} y_1^r \right] = \frac{n}{2} \sigma_i (y_1^i)^2 \end{aligned}$$

represent, from a geometrical point of view, a *spray* on the 1-jet space $J^1(\mathbb{R}, M^n)$.

Therefore, the *canonical nonlinear connection* associated to this spray has the local components (see [4])

$$M_{(1)1}^{(i)} \stackrel{def}{=} 2H_{(1)1}^{(i)} = -\kappa_{11}^1 y_1^i, \quad N_{(1)j}^{(i)} \stackrel{def}{=} \frac{\partial G_{(1)1}^{(i)}}{\partial y_1^j} = n\sigma_i y_1^i \delta_j^i.$$

■

3 The Cartan canonical Γ -linear connection. Its d-torsions and d-curvatures

The nonlinear connection (2.3) produces the dual *adapted bases* of d-vector fields (no sum by i)

$$\left\{ \frac{\delta}{\delta t} = \frac{\partial}{\partial t} + \kappa_{11}^1 y_1^p \frac{\partial}{\partial y_1^p} ; \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - n\sigma_i y_1^i \frac{\partial}{\partial y_1^i} ; \frac{\partial}{\partial y_1^i} \right\} \subset \mathcal{X}(E) \quad (3.1)$$

and d-covector fields (no sum by i)

$$\{dt ; dx^i ; \delta y_1^i = dy_1^i - \kappa_{11}^1 y_1^i dt + n\sigma_i y_1^i dx^i\} \subset \mathcal{X}^*(E), \quad (3.2)$$

where $E = J^1(\mathbb{R}, M^n)$. The naturalness of the geometrical adapted bases (3.1) and (3.2) is coming from the fact that, via a transformation of coordinates (1.2), their elements transform as the classical tensors. Therefore, the description of all subsequent geometrical objects on the 1-jet space $J^1(\mathbb{R}, M^n)$ (e.g., the Cartan canonical linear connection, its torsion and curvature etc.) will be given in local adapted components. Consequently, by direct computations, we obtain the following geometrical result:

Proposition 3.1 *The Cartan canonical Γ -linear connection, produced by the (t, x) -conformal deformed Berwald-Moór metric (1.1), has the following adapted local components (no sum by i, j or k):*

$$C\Gamma^* = \left(\kappa_{11}^1, G_{j1}^k = 0, L_{jk}^i = n\delta_j^i \delta_k^i \sigma_i, C_{j(k)}^{i(1)} = \mathfrak{C}_{jk}^i \cdot \frac{y_1^i}{y_1^j y_1^k} \right), \quad (3.3)$$

where

$$\mathfrak{C}_{jk}^i = -\frac{2}{n^2} + \frac{\delta_j^i + \delta_k^i + \delta_{jk}}{n} - \delta_j^i \delta_k^i.$$

The adapted components of the Cartan canonical connection are given by the formulas (see [4])

$$\begin{aligned} G_{j1}^k &\stackrel{def}{=} \frac{g^{km}}{2} \frac{\delta g_{mj}^*}{\delta t} = 0, \\ L_{jk}^i &\stackrel{def}{=} \frac{g^{im}}{2} \left(\frac{\delta g_{jm}^*}{\delta x^k} + \frac{\delta g_{km}^*}{\delta x^j} - \frac{\delta g_{jk}^*}{\delta x^m} \right) = n \delta_j^i \delta_k^i \sigma_i, \\ C_{j(k)}^{i(1)} &\stackrel{def}{=} \frac{g^{im}}{2} \left(\frac{\partial g_{jm}^*}{\partial y_1^k} + \frac{\partial g_{km}^*}{\partial y_1^j} - \frac{\partial g_{jk}^*}{\partial y_1^m} \right) = \frac{g^{im}}{2} \frac{\partial g_{jm}^*}{\partial y_1^k} = \mathfrak{C}_{jk}^i \cdot \frac{y_1^i}{y_1^j y_1^k}, \end{aligned}$$

where we also used the equality

$$\frac{\delta g_{jm}^*}{\delta x^k} = n \delta_{jk} g_{jm}^* \sigma_k + n \delta_{mk} g_{jm}^* \sigma_k.$$

Remark 3.2 It is important to note that the vertical d-tensor $C_{j(k)}^{i(1)}$ also has the properties (see also [9], [13] and [14]):

$$C_{j(k)}^{i(1)} = C_{k(j)}^{i(1)}, \quad C_{j(m)}^{i(1)} y_1^m = 0, \quad C_{j(m)}^{m(1)} = 0, \quad C_{i(k)|m}^{m(1)} = 0, \quad (3.4)$$

with sum by m , where

$$C_{i(k)|j}^{l(1)} \stackrel{def}{=} \frac{\delta C_{i(k)}^{l(1)}}{\delta x^j} + C_{i(k)}^{r(1)} L_{rj}^l - C_{r(k)}^{l(1)} L_{ij}^r - C_{i(r)}^{l(1)} L_{kj}^r.$$

Proposition 3.3 The Cartan canonical connection of the (t, x) -conformal deformed Berwald-Moór metric (1.1) has **two** effective local torsion d-tensors:

$$\begin{aligned} R_{(1)ij}^{(r)} &= n (\delta_i^r \sigma_{rj} - \delta_j^r \sigma_{ri}) y_1^r, \\ P_{i(j)}^{r(1)} &= \left(-\frac{2}{n^2} + \frac{\delta_i^r + \delta_j^r + \delta_{ij}}{n} - \delta_i^r \delta_j^r \right) \cdot \frac{y_1^r}{y_1^i y_1^j}, \end{aligned}$$

where

$$\sigma_{pq} := \frac{\partial^2 \sigma}{\partial x^p \partial x^q}.$$

Proof. Generally, an h -normal Γ -linear connection on the 1-jet space $J^1(\mathbb{R}, M^n)$ has *eight* effective local d-tensors of torsion (for more details, see [4]). For the Cartan canonical connection (3.3) these reduce only to *two* (the other six are zero):

$$R_{(1)ij}^{(r)} \stackrel{def}{=} \frac{\delta N_{(1)i}^{(r)}}{\delta x^j} - \frac{\delta N_{(1)j}^{(r)}}{\delta x^i}, \quad P_{i(j)}^{r(1)} \stackrel{def}{=} C_{i(j)}^{r(1)}.$$

■

Proposition 3.4 *The Cartan canonical connection of the (t, x) -conformal deformed Berwald-Moór metric (1.1) has **three** effective local curvature d-tensors:*

$$R_{ijk}^l = \frac{\partial L_{ij}^l}{\partial x^k} - \frac{\partial L_{ik}^l}{\partial x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l + C_{i(r)}^{l(1)} R_{(1)jk}^{(r)}, \quad P_{ij(k)}^l = -C_{i(k)|j}^{l(1)},$$

$$S_{i(j)(k)}^{l(1)(1)} \stackrel{\text{def}}{=} \frac{\partial C_{i(j)}^{l(1)}}{\partial y_1^k} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y_1^j} + C_{i(j)}^{r(1)} C_{r(k)}^{l(1)} - C_{i(k)}^{r(1)} C_{r(j)}^{l(1)}.$$

Proof. Generally, an h -normal Γ -linear connection on the 1-jet space $J^1(\mathbb{R}, M^n)$ has *five* effective local d-tensors of curvature (for more details, see [4]). For the Cartan canonical connection (3.3) these reduce only to *three* (the other two are zero); these are

$$R_{ijk}^l \stackrel{\text{def}}{=} \frac{\delta L_{ij}^l}{\delta x^k} - \frac{\delta L_{ik}^l}{\delta x^j} + L_{ij}^r L_{rk}^l - L_{ik}^r L_{rj}^l + C_{i(r)}^{l(1)} R_{(1)jk}^{(r)},$$

$$P_{ij(k)}^l \stackrel{\text{def}}{=} \frac{\partial L_{ij}^l}{\partial y_1^k} - C_{i(k)|j}^{l(1)} + C_{i(r)}^{l(1)} P_{(1)j(k)}^{(r)} = -C_{i(k)|j}^{l(1)},$$

$$S_{i(j)(k)}^{l(1)(1)} \stackrel{\text{def}}{=} \frac{\partial C_{i(j)}^{l(1)}}{\partial y_1^k} - \frac{\partial C_{i(k)}^{l(1)}}{\partial y_1^j} + C_{i(j)}^{r(1)} C_{r(k)}^{l(1)} - C_{i(k)}^{r(1)} C_{r(j)}^{l(1)},$$

where we used the equality

$$P_{(1)j(k)}^{(r)(1)} \stackrel{\text{def}}{=} \frac{\partial N_{(1)j}^{(r)}}{\partial y_1^k} - L_{jk}^r = 0.$$

■

4 Field-like geometrical models associated to the (t, x) -conformal deformation of the Berwald-Moór metric

4.1 Gravitational-like geometrical model

The (t, x) -conformal deformed Berwald-Moór metric (1.1) produces on the 1-jet space $J^1(\mathbb{R}, M^n)$ the adapted metrical d-tensor (sum by i and j)

$$\mathbf{G} = h_{11} dt \otimes dt + g_{ij}^* dx^i \otimes dx^j + h^{11} g_{ij}^* \delta y_1^i \otimes \delta y_1^j, \quad (4.1)$$

where g_{ij}^* is given by (2.1), and we have

$$\delta y_1^i = dy_1^i - \kappa_{11}^1 y_1^i dt + n \sigma_i y_1^i dx^i \quad (\text{no sum by } i).$$

From an abstract physical point of view, the metrical d-tensor (4.1) may be regarded as a “*non-isotropic gravitational potential*”. In our geometric-physical

approach, one postulates that the non-isotropic gravitational potential \mathbf{G} is governed by the following *geometrical Einstein-like equations*:

$$\text{Ric} (CT)^* - \frac{\text{Sc} (CT)^*}{2} \mathbf{G} = \mathcal{K} \mathcal{T}, \quad (4.2)$$

where

- ◆ $\text{Ric} (CT)^*$ is the *Ricci d-tensor* associated to the Cartan canonical linear connection (3.3);
- ◆ $\text{Sc} (CT)^*$ is the *scalar curvature*;
- ◆ \mathcal{K} is the *Einstein constant* and \mathcal{T} is the *intrinsic non-isotropic stress-energy d-tensor of matter*.

Therefore, using the adapted basis of vector fields (3.1), we can locally describe the global geometrical Einstein-like equations (4.2). Consequently, some direct computations lead to:

Lemma 4.1 *The Ricci tensor of the Cartan canonical connection CT^* of the (t, x) -conformal deformed Berwald-Moór metric (1.1) has the following **two** effective local Ricci d-tensors (no sum by i, j, k or l):*

$$R_{ij} = \begin{cases} -\sigma_{ij} - \sum_{\substack{m=1 \\ m \neq j}}^n \sigma_{jm} \frac{y_1^m}{y_1^i}, & i \neq j \\ 0, & i = j, \end{cases} \quad (4.3)$$

$$S_{(i)(j)}^{(1)(1)} = \left[\frac{2}{n^2} - \frac{1}{n} + \left(1 - \frac{2}{n} \right) \delta_{ij} \right] \cdot \frac{1}{y_1^i y_1^j}.$$

Proof. Generally, the Ricci tensor of the Cartan canonical connection CT produced by an arbitrary jet Lagrangian function is determined by *six* effective local Ricci d-tensors (for more details, see [4]). For our particular Cartan canonical connection (3.3) these reduce only to the following *two* (the other four are zero):

$$\begin{aligned} R_{ij} &\stackrel{def}{=} R_{ijm}^m = \frac{\partial L_{ij}^m}{\partial x^m} - \frac{\partial L_{im}^m}{\partial x^j} + L_{ij}^r L_{rm}^m - L_{im}^r L_{rj}^m + C_{i(r)}^{m(1)} R_{(1)jm}^{(r)}, \\ S_{(i)(j)}^{(1)(1)} &\stackrel{def}{=} S_{i(j)(m)}^{m(1)(1)} = \frac{\partial C_{i(j)}^{m(1)}}{\partial y_1^m} - \frac{\partial C_{i(m)}^{m(1)}}{\partial y_1^j} + C_{i(j)}^{r(1)} C_{r(m)}^{m(1)} - C_{i(m)}^{r(1)} C_{r(j)}^{m(1)} = \\ &= \frac{\partial C_{i(j)}^{m(1)}}{\partial y_1^m} - C_{i(m)}^{r(1)} C_{r(j)}^{m(1)}, \end{aligned}$$

with sum by r and m . ■

Lemma 4.2 *The scalar curvature of the Cartan canonical connection $C\Gamma^*$ of the (t, x) -conformal deformed Berwald-Moór metric (1.1) has the value*

$$\text{Sc} (C\Gamma^*) = -e^{-2\sigma} G_{1[n]}^{-2/n} [4nY_{11} + (n^2 - 3n + 2) h_{11}],$$

where

$$Y_{11} = \sum_{\substack{p, q=1 \\ p < q}}^n \sigma_{pq} y_1^p y_1^q.$$

Proof. The scalar curvature of the Cartan canonical connection (3.3) is given by the formula (for more details, see [4])

$$\text{Sc} (C\Gamma^*) = {}^*g^{pq} R_{pq} + h_{11} {}^*g^{pq} S_{(p)(q)}^{(1)(1)}.$$

■

The local description in the adapted basis of vector fields (3.1) of the global geometrical Einstein-like equations (4.2) is given by (for more details, see [4]):

Proposition 4.3 *The **geometrical Einstein-like equations** produced by the (t, x) -conformal deformed Berwald-Moór metric (1.1) are locally described by:*

$$\left\{ \begin{array}{l} e^{-2\sigma} G_{1[n]}^{-2/n} \left[2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right] h_{11} = \mathcal{K} \mathcal{T}_{11} \\ R_{ij} + e^{-2\sigma} G_{1[n]}^{-2/n} \left[2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right] {}^*g_{ij} = \mathcal{K} \mathcal{T}_{ij} \\ S_{(i)(j)}^{(1)(1)} + e^{-2\sigma} G_{1[n]}^{-2/n} \left[2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right] h^{11} {}^*g_{ij} = \mathcal{K} \mathcal{T}_{(i)(j)}^{(1)(1)} \\ 0 = \mathcal{T}_{1i}, \quad 0 = \mathcal{T}_{i1}, \quad 0 = \mathcal{T}_{(i)1}^{(1)} \\ 0 = \mathcal{T}_{1(i)}^{(1)}, \quad 0 = \mathcal{T}_{i(j)}^{(1)}, \quad 0 = \mathcal{T}_{(i)j}^{(1)}. \end{array} \right. \quad (4.4)$$

Corollary 4.4 *The non-isotropic stress-energy d-tensor of matter \mathcal{T} satisfies the following **geometrical conservation laws** (sum by m):*

$$\left\{ \begin{array}{l} \mathcal{T}_{1/1}^1 + \mathcal{T}_{1|m}^m + \mathcal{T}_{(1)1}^{(m)}|_{(m)}^{(1)} = 0 \\ \mathcal{T}_{i/1}^1 + \mathcal{T}_{i|m}^m + \mathcal{T}_{(1)i}^{(m)}|_{(m)}^{(1)} = E_{i|m}^m \\ \mathcal{T}_{(i)/1}^{(1)} + \mathcal{T}_{(i)|m}^{m(1)} + \mathcal{T}_{(1)(i)}^{(m)(1)}|_{(m)}^{(1)} = \frac{2e^{-2\sigma} G_{1[n]}^{-2/n}}{\mathcal{K}} \cdot \left[n \frac{\partial Y_{11}}{\partial y_1^i} - 2 \frac{Y_{11}}{y_1^i} \right], \end{array} \right.$$

where (sum by r):

$$\mathcal{T}_1^1 \stackrel{def}{=} h^{11} \mathcal{T}_{11} = \mathcal{K}^{-1} e^{-2\sigma} G_{1[n]}^{-2/n} \left[2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right],$$

$$\mathcal{T}_1^m \stackrel{def}{=} g^{mr} \mathcal{T}_{r1} = 0, \quad \mathcal{T}_{(1)1}^{(m)} \stackrel{def}{=} h_{11} g^{mr} \mathcal{T}_{(r)1}^{(1)} = 0, \quad \mathcal{T}_i^1 \stackrel{def}{=} h^{11} \mathcal{T}_{1i} = 0,$$

$$\begin{aligned} \mathcal{T}_i^m \stackrel{def}{=} g^{mr} \mathcal{T}_{ri} &:= E_i^m = \mathcal{K}^{-1} \left[g^{mr} R_{ri} + \right. \\ &\quad \left. + e^{-2\sigma} G_{1[n]}^{-2/n} \left(2nY_{11} + \frac{n^2 - 3n + 2}{2} h_{11} \right) \delta_i^m \right], \end{aligned}$$

$$\mathcal{T}_{(1)i}^{(m)} \stackrel{def}{=} h_{11} g^{mr} \mathcal{T}_{(r)i}^{(1)} = 0, \quad \mathcal{T}_{(i)}^{1(1)} \stackrel{def}{=} h^{11} \mathcal{T}_{1(i)}^{(1)} = 0, \quad \mathcal{T}_{(i)}^{m(1)} \stackrel{def}{=} g^{mr} \mathcal{T}_{r(i)}^{(1)} = 0,$$

$$\begin{aligned} \mathcal{T}_{(1)(i)}^{(m)(1)} \stackrel{def}{=} h_{11} g^{mr} \mathcal{T}_{(r)(i)}^{(1)(1)} &= \frac{e^{-2\sigma} G_{1[n]}^{-2/n}}{\mathcal{K}} \cdot \left[\frac{n-2}{n} h_{11} \frac{y_1^m}{y_1^i} + \right. \\ &\quad \left. + \left(2nY_{11} + \frac{n^2 - 5n + 6}{2} h_{11} \right) \delta_i^m \right], \end{aligned}$$

and we also have (summation by m and r , but no sum by i)

$$\mathcal{T}_{1/1}^1 = \frac{\delta \mathcal{T}_1^1}{\delta t}, \quad \mathcal{T}_{1|m}^m \stackrel{def}{=} \frac{\delta \mathcal{T}_1^m}{\delta x^m} + \mathcal{T}_1^r L_{rm}^m,$$

$$\mathcal{T}_{(1)1}^{(m)}|_{(m)}^{(1)} \stackrel{def}{=} \frac{\partial \mathcal{T}_{(1)1}^{(m)}}{\partial y_1^m} + \mathcal{T}_{(1)1}^{(r)} C_{r(m)}^{m(1)} = \frac{\partial \mathcal{T}_{(1)1}^{(m)}}{\partial y_1^m},$$

$$\mathcal{T}_{i/1}^1 \stackrel{def}{=} \frac{\delta \mathcal{T}_i^1}{\delta t} + \mathcal{T}_i^1 K_{11}^1 - \mathcal{T}_r^1 G_{i1}^r = \frac{\delta \mathcal{T}_i^1}{\delta t} + \mathcal{T}_i^1 K_{11}^1,$$

$$\mathcal{T}_{i|m}^m \stackrel{def}{=} \frac{\delta \mathcal{T}_i^m}{\delta x^m} + \mathcal{T}_i^r L_{rm}^m - \mathcal{T}_r^m L_{im}^r = E_{i|m}^m := \frac{\delta E_i^m}{\delta x^m} + n E_i^m \sigma_m - n E_i^i \sigma_i,$$

$$\mathcal{T}_{(1)i}^{(m)}|_{(m)}^{(1)} \stackrel{def}{=} \frac{\partial \mathcal{T}_{(1)i}^{(m)}}{\partial y_1^m} + \mathcal{T}_{(1)i}^{(r)} C_{r(m)}^{m(1)} - \mathcal{T}_{(1)r}^{(m)} C_{i(m)}^{r(1)} = \frac{\partial \mathcal{T}_{(1)i}^{(m)}}{\partial y_1^m} - \mathcal{T}_{(1)r}^{(m)} C_{i(m)}^{r(1)},$$

$$\mathcal{T}_{(i)/1}^{1(1)} \stackrel{def}{=} \frac{\delta \mathcal{T}_{(i)}^{1(1)}}{\delta t} + 2 \mathcal{T}_{(i)}^{1(1)} K_{11}^1, \quad \mathcal{T}_{(i)|m}^{m(1)} \stackrel{def}{=} \frac{\delta \mathcal{T}_{(i)}^{m(1)}}{\delta x^m} + \mathcal{T}_{(i)}^{r(1)} L_{rm}^m - \mathcal{T}_{(r)}^{m(1)} L_{im}^r,$$

$$\mathcal{T}_{(1)(i)}^{(m)(1)}|_{(m)}^{(1)} \stackrel{def}{=} \frac{\partial \mathcal{T}_{(1)(i)}^{(m)(1)}}{\partial y_1^m} + \mathcal{T}_{(1)(i)}^{(r)(1)} C_{r(m)}^{m(1)} - \mathcal{T}_{(1)(r)}^{(m)(1)} C_{i(m)}^{r(1)} = \frac{\partial \mathcal{T}_{(1)(i)}^{(m)(1)}}{\partial y_1^m}.$$

Proof. The local Einstein-like equations (4.4), together with some direct computations, lead us to what we were looking for. Also note that we have (summation by m and r)

$$\mathcal{T}_{(1)(r)}^{(m)(1)} C_{i(m)}^{r(1)} = 0.$$

■

4.2 Electromagnetic-like geometrical model

In the book [4], a geometrical theory for electromagnetism was also created, using only a given Lagrangian function L on the 1-jet space $J^1(\mathbb{R}, M^n)$. In the background of the jet single-time (one-parameter) Lagrange geometry from [4], one works with the following *non-isotropic electromagnetic distinguished 2-form* (sum by i and j):

$$\mathbf{F} = F_{(i)j}^{(1)} \delta y_1^i \wedge dx^j,$$

where (sum by m and r)

$$F_{(i)j}^{(1)} = \frac{h^{11}}{2} \left[g_{jm}^* N_{(1)i}^{(m)} - g_{im}^* N_{(1)j}^{(m)} + \left(g_{ir}^* L_{jm}^r - g_{jr}^* L_{im}^r \right) y_1^m \right].$$

This is characterized by some natural *geometrical Maxwell-like equations* (for more details, see [10] and [4]).

Remark 4.5 *The Lagrangian function that governs the movement law of a particle of mass $m \neq 0$ and electric charge e , which is displaced concomitantly into an environment endowed both with a gravitational field and an electromagnetic one, is given by*

$$L(t, x^k, y_1^k) = mch^{11}(t) \varphi_{ij}(x^k) y_1^i y_1^j + \frac{2e}{m} A_{(i)}^{(1)}(t, x^k) y_1^i, \quad (4.5)$$

where

- the semi-Riemannian metric $\varphi_{ij}(x)$ represents the **isotropic gravitational potential**;
- $A_{(i)}^{(1)}(t, x)$ are the components of a d -tensor on the 1-jet space $J^1(\mathbb{R}, M^n)$ representing the **electromagnetic potential**.

Note that the jet Lagrangian function (4.5) is a natural extension of the Lagrangian (defined on the tangent bundle) used in electrodynamics by Miron and Anastasiei [10]. In our jet Lagrangian formalism applied to (4.5), the **electromagnetic-like components** become classical ones (see [4]):

$$F_{(i)j}^{(1)} = -\frac{e}{2m} \left(\frac{\partial A_{(i)}^{(1)}}{\partial x^j} - \frac{\partial A_{(j)}^{(1)}}{\partial x^i} \right).$$

Moreover, the second set of **geometrical Maxwell-like equations** reduce to the classical ones too (for more details, see [10], [4]):

$$\sum_{\{i,j,k\}} F_{(i)j|k}^{(1)} = 0,$$

where

$$F_{(i)j|k}^{(1)} = \frac{\partial F_{(i)j}^{(1)}}{\partial x^k} - F_{(m)j}^{(1)} \gamma_{ik}^m - F_{(i)m}^{(1)} \gamma_{jk}^m.$$

Also, the **geometrical Einstein-like equations** attached to the Lagrangian (4.5) (see [10], [4]) are the same with the famous classical ones (associated to the semi-Riemannian metric $\varphi_{ij}(x)$). In author's opinion, these facts suggest some kind of naturalness for the present abstract Lagrangian non-isotropic electromagnetic and gravitational geometrical theory.

Via some direct calculations, we easily deduce that the (t, x) -conformal deformed Berwald-Moór metric (1.1) produces null non-isotropic electromagnetic components:

$$F_{(i)j}^{(1)} = 0.$$

It follows that our (t, x) -conformal deformed jet Berwald-Moór geometrical electromagnetic-like theory is trivial. This fact probably suggests that the (t, x) -conformal deformed Berwald-Moór geometrical structure (1.1) has rather gravitational connotations than electromagnetic ones.

As a conclusion, it is possible for the recent Voicu-Siparov approach of the electromagnetism in spaces with anisotropic metrics (that electromagnetic approach is different from the electromagnetic theory exposed above, and it is developed in the paper [19]) to give other interesting electromagnetic-geometrical results for spaces endowed with the Berwald-Moór geometrical structure.

Open problem. The author of this paper believes that the finding of some possible real physical interpretations for the present non-isotropic Berwald-Moór geometrical approach of gravity and electromagnetism may be an open problem for physicists.

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